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Problem Set 6

1. Momentum Conservation: Another Collision, and C of M

A small ball of mass m is placed on top of a larger ball of mass M, and the two balls are dropped from a height h. Assume $m \ll M$, and all collisions are elastic. How high does the smaller ball go?

Ans: When the two balls hit the ground, they have the velocity

$$v = \sqrt{2gh}$$

pointing down to the earth. Since the mass of the earth is much larger than the mass of the larger ball, it bounces back with velocity $-\sqrt{2gh}$ and hits the smaller ball.

Considering the collision of the two balls in the larger ball's reference frame. The smaller ball has velocity $2\sqrt{2gh}$ with respect to the larger ball, and because $m \ll M$, after the collision it has velocity $-2\sqrt{2gh}$, while the momentum of the larger ball only changes by a negligible amount. Therefore the velocity of the smaller ball with respect to the lab frame is $-\sqrt{2gh} - 2\sqrt{2gh} = -3\sqrt{2gh}$, pointing up vertically.

The height the smaller ball finally goes to is thus

$$h' = \frac{(-3\sqrt{2gh})^2}{2g} = 9h$$

2. Gradients

The gravitational potential from a mass M at a point $\vec{r}$ in space is defined as

$$\phi(x, y, z) = -\frac{G_N M}{r}$$

where G_N is Newton's constant, and $r \equiv |\vec{r}| = \sqrt{x^2+y^2+z^2}$ The potential is defined such that a second mass m will be acted upon by a force given by

$$\vec{F}(\vec{r}) = -m\vec{\nabla}\phi$$

a. Find the expression for the force at the point $\vec{r} = (x, y, z)$.

b. Calculate the force between you and the student sitting closest to you in class. (Give the direction as well as the magnitude, and round all numbers so that you don't need a calculator.)

Ans: a. Since

$$\nabla\frac{1}{r} = -\frac{1}{r^2}\nabla r = -\frac{\hat{r}}{r^2}$$

we have

$$\vec{F}(\vec{r}) = -m\nabla\left(-\frac{G_N M}{r}\right)$$
$$= +G_N M m \nabla\frac{1}{r}$$
$$= -\frac{G_N M m}{r^2}\hat{r}$$

b. According to the expression of $\vec{F}(\vec{r})$, the force between me and the closest student is

$$F = \frac{G_N m_1 m_2}{r^2}$$
$$= \frac{6.67\times10^{-11}\ N\cdot m^2\cdot kg^{-2}\cdot(70kg)^2}{(0.5m)^2}$$
$$= 1.3\times10^{-6}\,N$$

and pointing toward each other.

3. Cross-Product and Identities Involving Del

(1) Prove:

$$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{C}\cdot\vec{A})\vec{B} - (\vec{B}\cdot\vec{A})\vec{C}$$

(2) Prove:

$$\nabla\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\nabla)\vec{A} - (\vec{A}\cdot\nabla)\vec{B} + (\nabla\cdot\vec{B})\vec{A} - (\nabla\cdot\vec{A})\vec{B}$$

(3) Prove:

$$\nabla\cdot(\nabla\times\vec{A}) = 0$$

Ans. (1) We know

$$(\vec{V}\times\vec{W})_i = \epsilon_{ijk}V_j W_k\ ,\quad \vec{V}\cdot\vec{W} = V_i W_i$$

and

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{il}$$ (ϵ_{ijk} as the Levi-Civita symbol)

using the Einstein's sum rule. Therefore

$$(\vec{A}\times(\vec{B}\times\vec{C}))_i = \epsilon_{ijk}A_j(\vec{B}\times\vec{C})_k$$
$$= \epsilon_{ijk}A_j\epsilon_{klm}B_l C_m$$
$$= \epsilon_{kij}\epsilon_{klm}A_j B_l C_m$$
$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})A_j B_l C_m$$
$$= A_m C_m B_i - A_l B_l C_i = ((\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C})_i$$

(2) $(\nabla\times(\vec{A}\times\vec{B}))_i$

$$= \epsilon_{ijk}\,\partial_j\,(\vec{A}\times\vec{B})_k$$
$$= \epsilon_{ijk}\,\partial_j\,(\epsilon_{klm}A_l B_m)$$
$$= \epsilon_{kij}\,\epsilon_{klm}\,\partial_j(A_l B_m)$$
$$= (\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})\,(\partial_j A_l B_m + \partial_j B_m A_l)$$
$$= B_m\partial_m A_i - B_i\partial_l A_l + A_i\partial_m B_m + (-A_l\partial_l B_i)$$
$$= ((\vec{B}\cdot\nabla)\vec{A}-(\vec{A}\cdot\nabla)\vec{B}+(\nabla\cdot\vec{B})\vec{A}-(\nabla\cdot\vec{A})\vec{B})_i$$

(3) $\nabla\cdot(\nabla\times\vec{A})$

$$= \partial_i\,(\nabla\times\vec{A})_i$$
$$= \partial_i\,\epsilon_{ijk}\,\partial_j A_k$$
$$= (\epsilon_{ijk}\,\partial_i\partial_j)\,A_k$$
$$= 0$$

4. Forces from Potentials

Find the forces for the following potential energies

a. $U = Ax^2 + By^2 + Cz^2$

b. $U = A\ln(x^2+y^2+z^2)$

c. $U = A\cos\theta/r^2$

Ans: a.
$$F_x = -\frac{\partial U}{\partial x} = -2Ax$$
$$F_y = -\frac{\partial U}{\partial y} = -2By$$
$$F_z = -\frac{\partial U}{\partial z} = -2Cz$$

b.
$$F_x = -\frac{\partial U}{\partial x} = -\frac{2Ax}{x^2+y^2+z^2}$$
$$F_y = -\frac{\partial U}{\partial y} = -\frac{2Ay}{x^2+y^2+z^2}$$
$$F_z = -\frac{\partial U}{\partial z} = -\frac{2Az}{x^2+y^2+z^2}$$

c.
$$F_r = -\frac{\partial U}{\partial r} = \frac{2A\cos\theta}{r^3}$$
$$F_\theta = -\frac{1}{r}\frac{\partial U}{\partial \theta} = \frac{A\sin\theta}{r^3}$$

5. Is My Force Conservative?

Determine whether each of these following forces is conservative. Find the potential energy function if it exists. A, α, β are constants.

a. $\vec{F} = A(3\hat{i} + z\hat{j} + y\hat{k})$

b. $\vec{F} = Axyz(\hat{i}+\hat{j}+\hat{k})$

c. $F_x = 3Ax^2y^5e^{\alpha z}$, $F_y = 5Ax^3y^4e^{\alpha z}$, $F_z = \alpha Ax^3y^5e^{\alpha z}$

d. $F_x = A\sin(\alpha y)\cos(\beta z)$, $F_y = -Ax\alpha\cos(\alpha y)\cos(\beta z)$, and $F_z = Ax\sin(\alpha y)\sin(\beta z)$

Ans: a. $\nabla\times\vec{F} = \hat{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \hat{j}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \hat{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$

$$= 0$$

So $\vec{F}$ is conservative. The potential energy

$$U(x,y,z) = -\int_{(0,0,0)}^{(x,y,z)} \vec{F}\cdot d\vec{r}$$

$$= -3Ax - Ayz$$

b. $\nabla\times\vec{F} = \hat{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \hat{j}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \hat{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$

$$= Ax(z-y)\,\hat{i} + Ay(x-z)\,\hat{j} + Az(y-x)\,\hat{k}$$

$$\neq 0$$

So $\vec{F}$ is not conservative.

c. $\nabla\times\vec{F} = \hat{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \hat{j}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \hat{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$

$$= 0$$

So $\vec{F}$ is conservative. The potential energy

$$U(x,y,z) = -\int_{(0,0,0)}^{(x,y,z)} \vec{F}\cdot d\vec{r}$$

$$= -Ax^3y^5e^{\alpha z}$$

d. $\nabla\times\vec{F} = \hat{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \hat{j}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \hat{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$

$$= A\alpha x(1-\beta)\cos(\alpha y)\sin(\beta z)\,\hat{i} - A(1+\beta)\sin(\alpha y)\sin(\beta z)\,\hat{j}$$

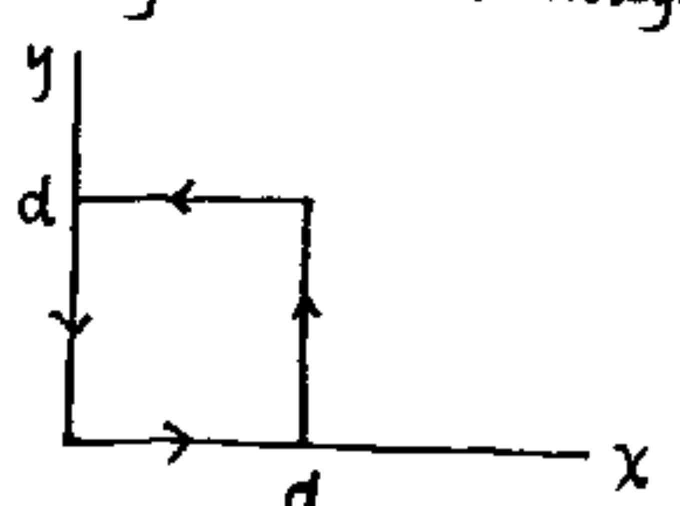
$$-2A\alpha\cos(\alpha y)\cos(\beta z)\,\hat{k}$$

$$\neq 0$$

So $\vec{F}$ is not conservative.

6. Calculating Work Along a Path.

How much work is done around the path that is shown by the force $\vec{F} = A(y^2\hat{i} + 2x^2\hat{j})$, where A is a constant and x and y are in meters? Find the answer by evaluating the line integral, and also by using Stokes' theorem.



Ans: The line integral

$$W = \oint \vec{F}\cdot d\vec{l}$$

$$= \int_0^d A\cdot 0^2\,dx + \int_0^d A\cdot 2d^2\,dy + \int_d^0 Ad^2\,dx + \int_d^0 A\cdot 2\cdot 0^2\,dy$$

$$= 0 + 2Ad^3 - Ad^3 + 0$$

$$= Ad^3$$

On the other hand,

$$\nabla \times \vec{F} = \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{k}$$
$$= (4Ax - 2Ay)\hat{k}$$

So we also have

$$W = \oint \vec{F} \cdot d\vec{l}$$
$$= \iint (\nabla \times \vec{F}) \cdot d\vec{S}$$
$$= \int_0^d dx \int_0^d dy \ (4Ax - 2Ay)$$
$$= \int_0^d dx \ (4Adx - Ad^2)$$
$$= 2Ad^3 - Ad^3$$
$$= Ad^3$$

which is exactly the same answer.