

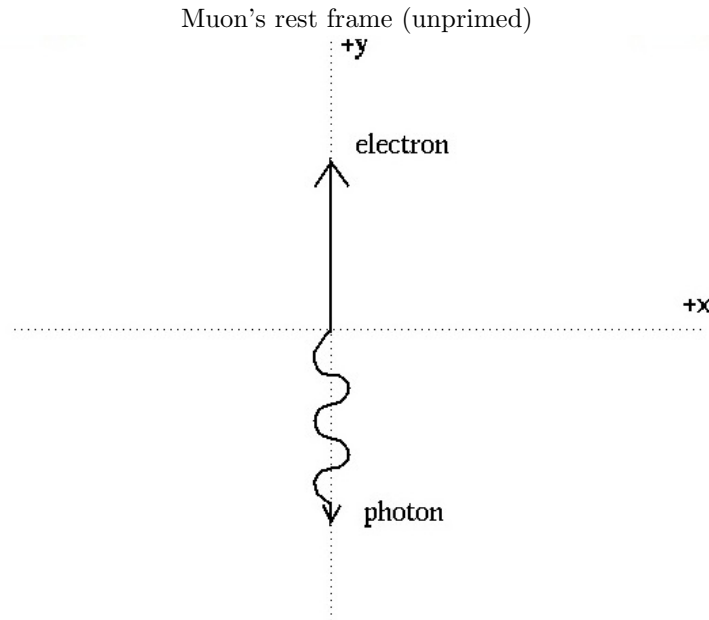
## Problem Set 5 Solutions

Due: November 3, 2008

Last modified: November 12, 2008

## Problem 1

(1.1)



(1.2)

In the muon's frame, the muon's 4-vector is

$$P_\mu^\mu = (M_\mu, 0, 0, 0)$$

where  $M_\mu = 100$  MeV is the given mass. It decays into the electron and photon, which have the form

$$\begin{aligned} P_{e^-}^\mu &= (E_{e^-}, 0, E_{e^-}, 0) \\ P_\gamma^\mu &= (E_\gamma, 0, -E_\gamma, 0) \end{aligned}$$

where the 3-momenta have the same magnitude as the energy since they are essentially massless ( $E^2 = p^2$ ). By conservation of 4-momentum, we get

$$\begin{aligned} M_\mu &= E_{e^-} + E_\gamma \\ 0 &= E_{e^-} - E_\gamma \end{aligned}$$

which gives

$$\begin{aligned} P_{e^-}^\mu &= (M_\mu/2, 0, M_\mu/2, 0) = (50, 0, 50, 0) \text{ MeV} \\ P_\gamma^\mu &= (M_\mu/2, 0, -M_\mu/2, 0) = (50, 0, -50, 0) \text{ MeV} \end{aligned}$$

### (1.3)

We just apply the usual Lorentz transformation to get to the lab frame. Ignoring the  $z$  direction as irrelevant, we get

$$P_{e^-}^{\mu'} = \begin{pmatrix} \gamma & \beta\gamma & 0 \\ \beta\gamma & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} M_\mu/2 \\ 0 \\ M_\mu/2 \end{pmatrix} = \begin{pmatrix} \gamma M_\mu/2 \\ \gamma\beta M_\mu/2 \\ M_\mu/2 \end{pmatrix}$$

The photon's 4-vector is the same, except the last entry has a negative sign:

$$P_\gamma^{\mu'} = \begin{pmatrix} \gamma & \beta\gamma & 0 \\ \beta\gamma & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} M_\mu/2 \\ 0 \\ -M_\mu/2 \end{pmatrix} = \begin{pmatrix} \gamma M_\mu/2 \\ \gamma\beta M_\mu/2 \\ -M_\mu/2 \end{pmatrix}$$

## Problem 2

We will split the motion up into the  $x$  (horizontal) and  $y$  (vertical) directions. If we call the initial velocity of the roustabout  $v_0$  and say  $\theta = \pi/4$ , then the  $x$  and  $y$  positions as functions of time are

$$\begin{aligned} x &= (v_0 \cos \theta)t \\ y &= (v_0 \sin \theta)t - \frac{1}{2}gt^2 \end{aligned}$$

From the first equation, we see that the time it will take to go a horizontal distance  $L$  is  $t_L = L/(v_0 \cos \theta)$ . Since the second equation is quadratic, we know that there are two times when she will be at ground level. The first is  $t = 0$  (when  $x = 0$ ), and the second must correspond to  $t_L$  (when  $x = L$ ). That is,

$$\begin{aligned} 0 &= (v_0 \sin \theta)t_L - \frac{1}{2}gt_L^2 \\ v_0 \sin \theta &= \frac{1}{2}g\left(\frac{L}{v_0 \cos \theta}\right) \\ v_0^2 &= \frac{gL}{2 \sin \theta \cos \theta}. \end{aligned}$$

Setting this expression aside for the moment, we can note that the highest point,  $h$ , will occur when she is halfway through her flight - at time  $t = t_L/2$ . Plugging into our equation of motion in the  $y$  direction gives

$$h = (v_0 \sin \theta)(t_L/2) - \frac{1}{2}g(t_L/2)^2 = \frac{v_0 L \sin \theta}{2v_0 \cos \theta} - \left(\frac{g}{2}\right)\left(\frac{L^2}{4v_0^2 \cos^2 \theta}\right)$$

Since  $v_0$  is not given in the problem, it cannot be part of our answer. Happily, we found an expression for  $v_0^2$  above, which can simply be substituted in:

$$h = \frac{L}{2} \tan \theta - \left(\frac{g}{2}\right)\left(\frac{L^2}{4 \cos^2 \theta}\right)\left(\frac{2 \sin \theta \cos \theta}{gL}\right) = \frac{L}{4} \tan \theta$$

Plugging in  $\theta = \pi/4$  gives us

$$\boxed{h = L/4}$$

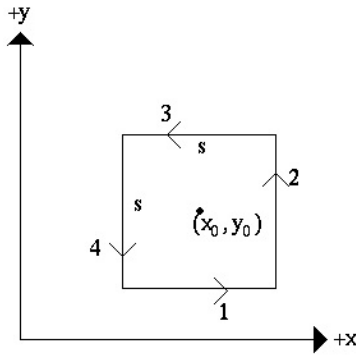


Figure 1: The paths used in the line integral for Problem 3.

### Problem 3

We are working with the function  $\vec{F} = z^2\hat{x} + x^2\hat{y} - y^2\hat{z}$ , where  $\hat{x}$ ,  $\hat{y}$ , and  $\hat{z}$  are unit vectors in the  $x$ ,  $y$ , and  $z$  directions. Since the path we are integrating over is in the  $x - y$  plane, we have  $z = 0$  everywhere, and can thus safely write  $\vec{F} = \vec{F}(x, y) = x^2\hat{y} - y^2\hat{z}$ .

We will divide the contour  $C$  up into four parts, as shown in Figure 1. This means we can write

$$\begin{aligned} \oint_C \vec{F}(x, y) \cdot \hat{t} ds &= \int_1 \vec{F}(x, y = y_0 - s/2) \cdot \hat{x} dx + \int_2 \vec{F}(x = x_0 + s/2, y) \cdot \hat{y} dy \\ &+ \int_3 \vec{F}(x, y = y_0 + s/2) \cdot (-\hat{x}) dx + \int_4 \vec{F}(x = x_0 - s/2, y) \cdot (-\hat{y}) dy \end{aligned}$$

Note that in the last two integrals, the path is going in the  $-\hat{x}$  and  $-\hat{y}$  directions.

Since  $\vec{F} \cdot \hat{x} = 0$  when  $z = 0$ , the first and third integrals vanish. This leaves us with

$$\begin{aligned} \oint_C \vec{F}(x, y) \cdot \hat{t} ds &= \int_{y_0 - s/2}^{y_0 + s/2} (x_0 + s/2)^2 dy - \int_{y_0 - s/2}^{y_0 + s/2} (x_0 - s/2)^2 dy \\ &= (x_0 + s/2)^2 s - (x_0 - s/2)^2 s \\ &= 2x_0 s \end{aligned}$$

If we wanted, we could verify that this is correct by using Stokes's Theorem. Briefly, we have

$$\int (\vec{\nabla} \times \vec{F}) \cdot \hat{n} dA = \int_{y_0 - s/2}^{y_0 + s/2} \int_{x_0 - s/2}^{x_0 + s/2} (\vec{\nabla} \times \vec{F}) \cdot \hat{z} dx dy = \int_{y_0 - s/2}^{y_0 + s/2} \int_{x_0 - s/2}^{x_0 + s/2} (2x) dx dy = 2x_0 s,$$

which agrees.

### Problem 4

We will solve this problem by using the fact that the work done by a conservative force is just the change in the potential energy. The potential energy of the box is given by  $U = mgh$ , with  $h$  the height of the center of mass.

I will call the height of the box  $L_1$  and the widths both  $L_2$ . See Figure 2.

We can see that in both cases, the center of mass is half the height of the highest point on the box. In the first case, where the box is sitting normally, this means the height is just  $h_1 = L_1/2$ . In

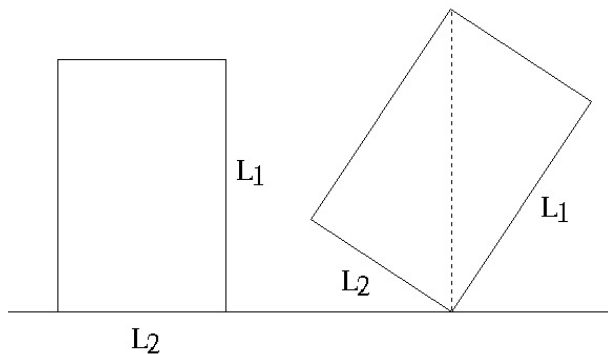


Figure 2: The box from Problem 4 in its initial position (left) and final tilted position (right).

the second case, we can use the Pythagorean Theorem to find that the height is the length of the diagonal, so  $h_2 = \sqrt{L_1^2 + L_2^2}/2$ . Then

$$W = mg(h_2 - h_1) = \frac{mg}{2} \left( \sqrt{L_1^2 + L_2^2} - L_1 \right)$$

Plugging in  $m = 100$  kg,  $g = 9.8$  m/s<sup>2</sup>,  $L_1 = 2$  m and  $L_2 = 1$  m gives us a value of  $W \approx 116$  J.

## Problem 5

### (5a)

I will call the mass of both pucks  $m$ .  $\vec{v}_0$  and  $\vec{v}_1$  will be the velocity of the initially moving puck, before and after the collision.  $\vec{v}_2$  will be the velocity of the other puck after the collision. Conservation of momentum tells us that the initial momentum of the puck system is equal to its final momentum. That is,  $m\vec{v}_0 = m\vec{v}_1 + m\vec{v}_2$ . We will assume that this is an elastic collision, so that energy is conserved. Here the energy is kinetic,  $E = \frac{1}{2}mv^2$ . This gives us  $mv_0^2/2 = mv_1^2/2 + mv_2^2/2$ . All of the factors of  $m$  and  $1/2$  cancel, leaving us with

$$\begin{aligned} \vec{v}_0 &= \vec{v}_1 + \vec{v}_2 \\ v_0^2 &= v_1^2 + v_2^2 \end{aligned}$$

### (5b)

To combine the above equations, simply square the first (since it is a vector equation, by this I mean to take the dot product of the first equation with itself) and subtract the second from it. That is,

$$\begin{aligned} v_0^2 - v_0^2 &= (v_1^2 + 2\vec{v}_1 \cdot \vec{v}_2 + v_2^2) - (v_1^2 + v_2^2) \\ 0 &= \vec{v}_1 \cdot \vec{v}_2 \end{aligned}$$

The above equation tells us that the dot product of the two final velocities must vanish - that is, they will be directed at right angles to each other. Since we know one moves at 30 degrees, the other must move at 60 degrees.

## Problem 6

Let's say that the ball is moving at velocity  $v_b$  and the truck is going at velocity  $v_t$ . Primed variables will denote velocities after the collision.

This problem is easiest to work in the frame of the truck. We don't need to do a full Lorentz Transformation here, since the velocities are so small - we can simply subtract off the velocity of the truck from all velocities. This will give us a ball velocity in the truck's frame of  $u_b = v_b - v_t$ . The truck is standing still in this frame, so it is as if the ball is elastically bouncing off of an immovable wall. That is, it will exit with the opposite velocity,  $u_b' = -u_b = v_t - v_b$ . Now we just need to transform back out of the truck's frame, by adding  $v_t$ :  $v_b' = 2v_t - v_b$ . With the numbers given, this works out to 0; the ball will stop moving horizontally.

## Problem 7

As in Figure 3 (left), let's define the angle  $\theta = 0$  to be the place where the tailer's end of the line meets the capstan, and  $\theta = \theta_m$  be the maximum angle where the rope is still in contact with the capstan before going to the sail.

Figure 3 (right) shows the free body diagram for a small segment of the rope, at an angle  $\theta$  around the capstan. This segment has an angular width of  $\Delta\theta$ . There are four forces acting on it (gravity is perpendicular to the page, and does not play a role in this problem). First, there is the small normal force  $\Delta N$  from the capstan on the rope, going in the radial direction. There is also a frictional force  $f$  acting on the point in question. Since the rope wants to slip toward the sail (larger  $\theta$ ), the friction wants to act in the opposite direction, toward smaller values of  $\theta$ . The other two forces are tensions. Though they act on the same piece of rope as the friction and normal, they act in the direction of the tangent at the end of the small segment rather than the direction of the tangent at the middle. Through geometric arguments (see Figure 4), we can see that the angle each of these makes with respect to the tangent in the middle is  $\Delta\theta/2$ .

Since the sailor is holding the rope so that it doesn't slip, we know that this segment of the rope is not accelerating and thus the net force on it is zero. It is easiest to separate the forces into the tangential and radial directions. Lets look first in the radial direction:

$$\sum F_r = 0 = \Delta N - T(\theta) \sin(\Delta\theta/2) - T(\theta + \Delta\theta) \sin(\Delta\theta/2)$$

$$\Delta N = \sin(\Delta\theta/2)(T(\theta) + T(\theta + \Delta\theta)).$$

Now we can look at the forces in the tangential direction:

$$\sum F_t = 0 = T(\theta + \Delta\theta) \cos(\Delta\theta/2) - f - T(\theta) \cos(\Delta\theta/2).$$

Since we know that at the moment before slipping, the frictional force is the product of the coefficient of friction and the normal force, we can rewrite this as

$$\mu \Delta N = \cos(\Delta\theta/2)(T(\theta + \Delta\theta) - T(\theta)).$$

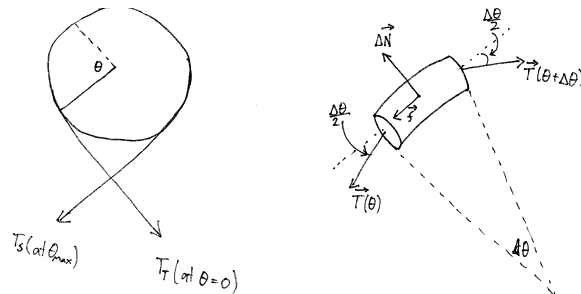


Figure 3: Problem 7 setup

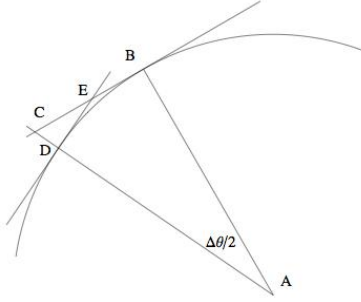


Figure 4: The angle between the tangents at the middle and ends of the segment is angle  $CED$ . We can find it by noting that triangles  $ABC$  and  $EDC$  are similar.

Substituting in our result from the radial direction gives

$$\mu \sin(\Delta\theta/2)(T(\theta + \Delta\theta) + T(\theta)) = \cos(\Delta\theta/2)(T(\theta + \Delta\theta) - T(\theta))$$

Now let's take  $\Delta\theta$  to be very small. This lets us approximate  $\sin(\Delta\theta/2) \approx \Delta\theta/2$  and  $\cos(\Delta\theta/2) \approx 1$ , and we can Taylor expand  $T$  to first order:

$$\frac{\mu\Delta\theta}{2} \left( T(\theta) + \Delta\theta \frac{dT}{d\theta} + T(\theta) \right) = T(\theta) + \Delta\theta \frac{dT}{d\theta} - T(\theta)$$

Ignoring terms of order  $\Delta\theta^2$  and simplifying, this reduces to the simple differential equation

$$\mu T = \frac{dT}{d\theta}.$$

This gives us

$$\begin{aligned} \mu \int_0^{\theta_m} d\theta &= \int_{T(\theta=0)}^{T(\theta=\theta_m)} \frac{dT}{T} \\ \mu\theta_m &= \log \frac{T(\theta_m)}{T(0)} \end{aligned}$$

Identifying  $T(0) = T_T$  and  $T(\theta_m) = T_S$ , we have

$$T_T = T_S e^{-\mu\theta_m}$$

## Problem 8

Define the total length of the chain to be  $\ell$ , and the length that is currently resting on the table to be  $x$  (that is, a length  $x$  has already fallen). We will assume that the chain has total mass  $m$ , and a uniform mass density  $\lambda = m/\ell$ . Note that since this density is constant, we also have  $dm = \lambda dx$  for a small portion of mass  $dm$  of the chain as it hits the table.

Let's consider the momentum of such a small piece of mass, just before it hits the table. It will have fallen a total length  $x$  under the influence of gravity, so its velocity will be  $v^2 = 2gx$ . It has mass  $dm$ , so the total momentum this small piece has is

$$dP = (dm)v$$

After the collision it has zero momentum since it stops moving, so the force it will exert on the table over the small time  $dt$  it takes to stop moving is

$$Fdt = (dm)v - 0$$

$$F = \frac{dm}{dt}v = \lambda \frac{dx}{dt}v = \lambda v^2 = 2\lambda gx = 2mg(x/\ell)$$

This is the force the falling part of the chain exerts on the table as it hits. Of course, we also need to consider the weight of the chain already resting on the table. This is just

$$F_w = \lambda gx = mg(x/\ell)$$

Thus the total force the table feels is

$$\boxed{F_{\text{tot}} = 3mg(x/\ell)}$$

If we wanted, we could instead use the product rule in Newton's Second Law:

$$F = \frac{dp}{dt} = \frac{d(mv)}{dt} = m \frac{dv}{dt} + \frac{dm}{dt}v$$

We have to be careful here about what we mean when we refer to "p" here, though - are we talking about the whole chain, the part of the chain on the table, or the part of the chain in the air? I find it simplest to refer to the part of the chain which is still falling. In this case,  $m = \lambda(\ell - x)$ , and  $dv/dt = g$ . The mass of the chain in the air is getting *smaller*, so we need to have a negative sign:  $dm/dt = -\lambda(dx/dt)$ . The velocity is the same as it was above,  $v = \sqrt{2gx}$ .

This gives us

$$F = \lambda g(\ell - x) - \lambda v^2 = \lambda g(\ell - x) - 2\lambda gx$$

This is the total force acting on the falling chain. We know that gravity is acting on the falling chain with force  $\lambda g(\ell - x)$ , so the remaining amount,  $-2\lambda gx$ , must be the force the table is exerting on this falling portion. The force the falling part is exerting on the table is the Newton's 3rd Law pair of this,  $+2\lambda gx$ . The total force on the table is this plus the force the resting portion of the chain is exerting,  $\lambda gx$ . In total, this gives us the same result as before,

$$\boxed{F_{\text{tot}} = 3mg(x/\ell)}$$