Physics 141 Autumn 2008

Problem Set 3 Solutions

Due: October 20, 2008 *Last modified: October 14, 2008*

Problem 1 (K&K 1.2)

We know that

$$\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\theta$$

with θ the angle between $\mathbf{A}$ and $\mathbf{B}$. Rearranging,

$$\cos\theta = \frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}||\mathbf{B}|} = \frac{A_xB_x + A_yB_y + A_zB_z}{\sqrt{A_x^2+A_y^2+A_z^2}\sqrt{B_x^2+B_y^2+B_z^2}}.$$

Plugging in numbers gives

$$\boxed{\cos\theta = -0.806}$$

Problem 2 (K&K 1.6)

Referring to Figure 1, the Law of Sines states

$$\frac{|\mathbf{A}|}{\sin\theta_A} = \frac{|\mathbf{B}|}{\sin\theta_B} = \frac{|\mathbf{C}|}{\sin\theta_C}. \tag{1}$$

To prove this, we will use two properties of the cross product. First, we take advantage of the fact that the cross product of two vectors $\mathbf{X}$ and $\mathbf{Y}$ is equal in magnitude to the area of the parallelogram whose sides have lengths $|\mathbf{X}|$ and $|\mathbf{Y}|$. Second, we use $|\mathbf{X}\times\mathbf{Y}| = |\mathbf{X}||\mathbf{Y}|\sin\theta$, with θ the angle between $\mathbf{X}$ and $\mathbf{Y}$.

The area A of the triangle can then be written in any of the following ways:

$$A = \frac{1}{2}|\mathbf{A}\times\mathbf{B}| = \frac{1}{2}|\mathbf{B}\times\mathbf{C}| = \frac{1}{2}|\mathbf{C}\times\mathbf{A}|.$$

Expanding the cross product gives

$$|\mathbf{A}||\mathbf{B}|\sin\theta_C = |\mathbf{B}||\mathbf{C}|\sin\theta_A = |\mathbf{C}||\mathbf{A}|\sin\theta_{\mathbf{B}}$$

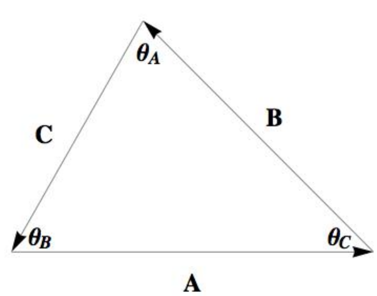

Figure 1: The triangle referred to in Problem 2

Dividing by $|\mathbf{A}|$ gives

$$|\mathbf{B}| \sin\theta_{\mathbf{C}} = |\mathbf{C}| \sin\theta_{\mathbf{B}} \Rightarrow \frac{|\mathbf{B}|}{\sin\theta_{\mathbf{B}}} = \frac{|\mathbf{C}|}{\sin\theta_{\mathbf{C}}}$$

Of course, we could have chosen to divide by $|\mathbf{B}|$ or $|\mathbf{C}|$ instead. Combining these gives us all of the equalities in Equation 1.

Problem 3 (K&K 1.7)

We can picture a right triangle formed by a line of length x along the horizontal axis, the unit vector $\hat{\mathbf{a}}$, and the vertical line of length y from $\hat{\mathbf{a}}$ to the horizontal axis. Since $\hat{\mathbf{a}}$ is a unit vector the hypotenuse has length one, and we can write $\cos\theta = x/1 = x$, and $\sin\theta = y/1 = y$. Thus we can write $\hat{\mathbf{a}} = (\cos\theta, \sin\theta)$. We can do the same thing to see $\hat{\mathbf{b}} = (\cos\phi, \sin\phi)$.

The angle between the two vectors is $|\theta - \phi|$, so taking their dot product gives

$$|\hat{\mathbf{a}} \cdot \hat{\mathbf{b}}| = |\hat{\mathbf{a}}||\hat{\mathbf{b}}| \cos(\theta - \phi).$$

Since cosine is an even function, it doesn't matter if we write $\cos(\theta - \phi)$ or $\cos(\phi - \theta)$. Evaluating the dot product and using $|\hat{\mathbf{a}}| = |\hat{\mathbf{b}}| = 1$, we see

$$\boxed{\cos\theta\cos\phi + \sin\theta\sin\phi = \cos(\theta - \phi)}$$

Problem 4 (K&K 1.8)

We can find a vector perpendicular to $\mathbf{A}$ and $\mathbf{B}$ by simply taking their cross product. However, we want to find a unit vector. We can do this by just dividing the cross product by its length, which does not change its direction. Thus we want

$$\hat{\mathbf{n}} = \frac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A} \times \mathbf{B}|}$$

Using

$$\mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y, A_z B_x - A_x B_z, A_x B_y - A_y B_x)$$

and plugging in numbers, we get

$$\hat{\mathbf{n}} = \frac{(2, -5, -3)}{\sqrt{(2)^2 + (-5)^2 + (-3)^2}}$$

$$\boxed{\hat{\mathbf{n}} = \frac{(2, -5, -3)}{\sqrt{38}}.}$$

Note that $-\hat{\mathbf{n}}$ is also perpendicular to $\mathbf{A}$ and $\mathbf{B}$.

Problem 5 (K&K 1.13)

Let v be the velocity with which the elevator ascends. Then at time T_1, the height of the elevator is $h = vT_1$. When the marble is dropped, it has initial velocity v and height h, and takes time T_2 to get back to the floor. Using the kinematic equation for position as a function of time,

$$0 = h + vT_2 - \frac{1}{2} g T_2^2.$$

Substituting in our expression for h gives

$$\begin{aligned} 0 &= vT_1 + vT_2 - \frac{1}{2}gT_2^2 \\ v &= \frac{g}{2}\frac{T_2^2}{T_1 + T_2} \end{aligned}$$

We can now use $h = vT_1$ to get

$$\boxed{h = \frac{g}{2}\frac{T_1 T_2^2}{T_1 + T_2}}$$

Problem 6 (K&K 1.21)

We define a coordinate system with origin at the peak of the hill. The x and y axes are horizontal and vertical, respectively. We can write down both the path of the rock and the path of the hill's surface in terms of x and y. Their intersection is the point where the rock will land. Algebraically, these conditions are

$$\begin{aligned} x &= (v_0 \cos\theta)t \\ y &= (v_0 \sin\theta)t - \frac{1}{2}gt^2 \\ y &= -x\tan\phi \end{aligned}$$

where v_0 is the speed the rock is thrown at. From the first of those equations, we get

$$t = \frac{x}{v_0 \cos\theta}.$$

Substituting that and the third equation into the second gives

$$-x\tan\phi = \frac{v_0 \sin\theta}{v_0\cos\theta}x - \frac{1}{2}g\frac{x^2}{v_0^2\cos^2\theta}$$

$$x = \frac{2v_0^2}{g}\Big[\tan\phi\cos^2\theta + \sin\theta\cos\theta\Big] \text{ or } x = 0.$$

We will ignore the $x = 0$ solution; it represents the starting position of the rock, and so it is not what we are looking for. Using trig identities, the solution we keep becomes

$$x = \frac{v_0^2}{g}\Big[\big(1 + \cos(2\theta)\big)\tan\phi + \sin(2\theta)\Big]$$

To maximize x, we will take its derivative with respect to θ, and then find the θ which makes this zero.

$$\frac{dx}{d\theta} = 0 = \frac{v_0^2}{g}\Big[-2\tan\phi\sin(2\theta) + 2\cos(2\theta)\Big]$$

$$\tan(2\theta) = \cot\phi$$

$$\theta = \frac{1}{2}\arctan\cot\phi.$$

If we now make the (rather reasonable) assumption that $0 \le \phi < \pi/2$, then we get

$$\boxed{\theta = \frac{1}{2}\big(\pi/2 - \phi\big)}$$