| **Physics 141** | Autumn 2008 |
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# Problem Set 2 Solutions

*Due: October 13, 2008* — *Last modified: October 15, 2008*

## Problem 1

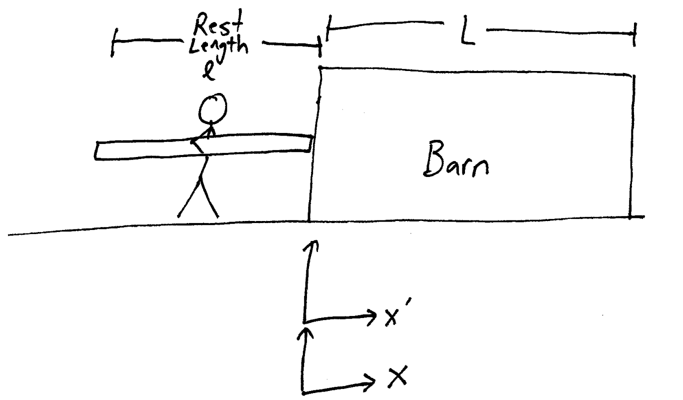


Figure 1: Picture for problem 1.1

### (1.1)

See Figure 1. In their own frames of reference, the pole is defined to have length $\ell$ and the barn has length $L$. A mutual origin is defined at the event where the front (right) of the pole meets the front (left) of the barn. All the motion is happening in the $x$ direction, which is positive to the right.

### (1.2)

The position of the front end of the pole will remain at 0 in the pole's / runner's reference frame, while the back will remain at $-\ell$. The time component of the four vectors will be the distance that needs to be travelled divided by the speed $\beta$ of the runner. Note that as the $y$ and $z$ components will remain the same throughout this problem, I am not going to bother with them, and write the four vectors as just $x^\mu = (t, x)$.

**(a):** $X_A^\mu = (0, 0)$.

**(b):** $X_B^\mu = (\frac{L}{\gamma\beta}, 0)$, since the length of the barn in this frame has been Lorentz contracted by a factor of $\gamma$.

**(c):** $X_C^\mu = (\frac{\ell}{\beta}, -\ell)$, since the pole is of length $\ell$.

**(d):** $X_D^\mu = (\frac{\ell}{\beta} + \frac{L}{\gamma\beta}, -\ell)$, since it needs to go both the length of the pole and the length of the barn.

## (1.3)

We can use the same reasoning here, using the fact that in the barn's frame, the front is always at 0 and the back is at $L$. The length of the pole here is Lorentz contracted to $\ell/\gamma$. We get

$$\begin{aligned} X_A^{\mu\prime} &= (0,0) \\ X_B^{\mu\prime} &= \Big(\frac{L}{\beta}, L\Big) \\ X_C^{\mu\prime} &= \Big(\frac{\ell}{\gamma\beta}, 0\Big) \\ X_D^{\mu\prime} &= \Big(\frac{\ell}{\gamma\beta} + \frac{L}{\beta}, L\Big). \end{aligned}$$

Alternatively, we could simply do a Lorentz contraction on the four vectors from the previous part to write them in the barn's frame:

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}.$$

When you do this, you can compute

$$\begin{aligned} X_A^{\mu\prime} &= (0,0) \\ X_B^{\mu\prime} &= \Big(\frac{L}{\beta}, L\Big) \\ X_C^{\mu\prime} &= \Big(\frac{\ell\gamma}{\beta} - \ell\beta\gamma, 0\Big) \\ &= \Big(\frac{\ell\gamma}{\beta}(1-\beta^2), 0\Big) \\ &= \Big(\frac{\ell}{\gamma\beta}, 0\Big) \\ X_D^{\mu\prime} &= \Big(\frac{L}{\beta} + \frac{\ell\gamma}{\beta} - \ell\beta\gamma, L\Big) \\ &= \Big(\frac{\ell}{\gamma\beta} + \frac{L}{\beta}, L\Big). \end{aligned}$$

which gives the same result that we got by doing it the other way, but with more algebra.

## (1.4)

If we assume that $\ell$ is exactly 20 feet long, and the barn is exactly 30 feet long, then when we plug in numbers we get (in order of time coordinate)

$$\begin{aligned} X_A^{\mu} &= (0,0)\ \text{ft} \\ X_B^{\mu} &= (30/\sqrt{3}, 0)\ \text{ft} \\ X_C^{\mu} &= (40/\sqrt{3}, -20)\ \text{ft} \\ X_D^{\mu} &= (70/\sqrt{3}, -20)\ \text{ft} \end{aligned}$$

### (1.5)

Doing the same for the barn's frame, we get a different order:

$$\begin{aligned} X_A^{\mu\,\prime} &= (0,0) \text{ ft} \\ X_C^{\mu\,\prime} &= (20/\sqrt{3},0) \text{ ft} \\ X_B^{\mu\,\prime} &= (60/\sqrt{3},30) \text{ ft} \\ X_D^{\mu\,\prime} &= (80/\sqrt{3},30) \text{ ft} \end{aligned}$$

### (1.6)

By looking at the ordering of events, we can conclude that according to the runner, first the front of the pole goes in and out of the barn, and then the back of the pole goes in and out. Thus the runner never sees the whole pole in the barn at the same time.

From the barn's perspective, however, first the front of the pole enters the barn, and then the back of the pole enters the barn. Following that, the front of the pole leaves the barn, and then the back of the pole leaves. Thus, the barn sees the whole pole inside the barn at the same time.

Because we have no preferred inertial frames, we cannot choose either perspective to be "more correct" than the other.

## Problem 2

Frame $F'$ sees frame $F$ moving at $\beta_1$ (which we will associate with $\gamma_1$), so we can relate four vectors between them according to the Lorentz transformation

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma_1 & \gamma_1\beta_1 \\ \gamma_1\beta_1 & \gamma_1 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix},$$

where we are ignoring the $y$ and $z$ directions because they are boring and stay the same.

Likewise, we can relate quantities in $F''$ to quantities in $F'$ via $\beta_2$ and it's associated $\gamma_2$:

$$\begin{pmatrix} t'' \\ x'' \end{pmatrix} = \begin{pmatrix} \gamma_2 & \gamma_2\beta_2 \\ \gamma_2\beta_2 & \gamma_2 \end{pmatrix} \begin{pmatrix} t' \\ x' \end{pmatrix}.$$

Our goal is to find a way to relate quantities in $F''$ to those in $F$. We know we can do this by some transformation

$$\begin{pmatrix} t'' \\ x'' \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix},$$

but we don't know what $\beta$ and $\gamma$ are yet. To find these out, we combine the first two transformations:

$$\begin{aligned} \begin{pmatrix} t'' \\ x'' \end{pmatrix} &= \begin{pmatrix} \gamma_2 & \gamma_2\beta_2 \\ \gamma_2\beta_2 & \gamma_2 \end{pmatrix} \begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma_2 & \gamma_2\beta_2 \\ \gamma_2\beta_2 & \gamma_2 \end{pmatrix} \begin{pmatrix} \gamma_1 & \gamma_1\beta_1 \\ \gamma_1\beta_1 & \gamma_1 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} \\ &= \begin{pmatrix} \gamma_1\gamma_2(1+\beta_1\beta_2) & \gamma_1\gamma_2(\beta_1+\beta_2) \\ \gamma_1\gamma_2(\beta_1+\beta_2) & \gamma_1\gamma_2(1+\beta_1\beta_2) \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} \end{aligned}$$

Thus we see that

$$\begin{aligned} \gamma &= \gamma_1\gamma_2(1+\beta_1\beta_2) \\ \gamma\beta &= \gamma_1\gamma_2(\beta_1+\beta_2) \end{aligned}$$

Eliminating $\gamma$ and solving for $\beta$ gives

$$\beta = \frac{\beta_1+\beta_2}{1+\beta_1\beta_2}$$

## Problem 3

### (3.1)

Using the notation from problem 2, we can associate Earth with $F''$, the mother/rocket ship with $F'$, and the small ship with $F$. Thus $\beta_1 = 0.6$ and $\beta_2 = 0.9$. Using the formula for $\beta$ we derived in problem 2, we get

$$\begin{aligned}\beta &= \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2} \\ &= \frac{0.6 + 0.9}{1 + (0.6)(0.9)} \\ \beta &\approx 0.974\end{aligned}$$

### (3.2)

Here the light from the headlight corresponds to $\beta_1 = 1$, and the rocket ship gives $\beta = 0.9$. Then

$$\begin{aligned}\beta &= \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2} \\ &= \frac{1 + 0.9}{1 + (1)(0.9)} \\ \beta &= 1\end{aligned}$$

which is exactly what we should have expected: the speed of the light is the same in whatever frame you go to.

### (3.3)

In this case, the speeds are so slow that you can, to excellent approximation, just add them together normally:

$$v = v_1 + v_2 = 50\text{ ft/s} + 90\text{ ft/s} = 140\text{ ft/s}$$

## Problem 4

### (4.1)

In it's own rest frame, the momentum components are zero and the energy component is just the rest mass:

$$P^\mu = (M, 0, 0, 0).$$

### (4.2)

The quark $q$ and anti-quark $\bar{q}$ must have the same energy ($E_b$), and the same magnitude of 3-momenta ($p_b$) because of momentum conservation. The signs of the momenta must be opposites though, since in this frame the Higgs began with zero total momentum.

$$\begin{aligned}E_q^\mu &= (E_b, p_b, 0, 0) \\ E_{\bar{q}}^\mu &= (E_b, -p_b, 0, 0)\end{aligned}$$

where I have defined the $x$ axis to be pointing in the direction the quark goes.

### (4.3)

The initial energy $E_i$ was just $M$, the mass of the Higgs. After the decay, the energy $E_f$ is split equally among the quarks. This gives

$$\begin{aligned} E_i &= E_f \\ M &= E_b + E_b \\ E_b &= M/2 \end{aligned}$$

### (4.4)

In the quark's own rest frame, we know its momentum 4-vector is just

$$P_q^{\mu\prime} = (m, 0, 0, 0)$$

where $m$ is the rest mass. The invariant length of this $P_q^{\mu\prime}$ must be the same as the invariant length of the $P_q^{\mu}$ we found earlier

$$\begin{aligned} |P_q^{\mu\prime}|^2 &= |P_q^{\mu}|^2 \\ m^2 &= E_b^2 - p_b^2 \\ p_b^2 &= E_b^2 - m^2 \\ p_b &= \sqrt{M^2/4 - m^2} \end{aligned}$$

### (4.5)

Plugging in $M = 115$ GeV and $m = 4$ GeV, we get

$$p_b = 57.4 \text{ GeV}$$

## Extra Credit Problem 5 - the Twin 'Paradox'

*NOTE: For display purposes, I am going to change the numbers to this problem. This should make it easier to resolve different lines in the diagram. I will use $\gamma = 2$ rather than $\gamma = 100$. The ship will still travel for 1 year out in its rest frame, and emit signals at one year intervals. From Earth, however, I will emit a signal every* 0.2 *years. This will give a reasonable number of signals to draw.*

The ship flies out to the casino for one year. Ignoring the $x$ and $y$ directions as irrelevant, in its own frame it is at position $x_c^{\mu} = (1 \text{ yr}, 0)$. We will be drawing everything in the Earth frame, so we need to transform to position $x_c^{\mu\prime} = (t', x') = (\gamma, \beta\gamma)$ in units of years. It takes the same amount of time to go back, so the position of the return is $x_r^{\mu} = (2\gamma, 0)$ in the Earth's frame. Thus $2\gamma$ years have passed on Earth. The position of the ship's return in the ship's frame is of course $x_r^{\mu\prime} = (2 \text{ yr}, 0)$.

A space-time diagram is shown in Figure 2. We can construct it as follows: Draw the path of the ship as two lines, one from the origin to the casino at $x_c^{\mu\prime}$, and then from the casino to the return point at $x_r^{\mu\prime}$.

When a signal from Earth is emitted at time $\tau$, it travels along a null path until it gets to the ship. In units where $c = 1$, these null paths correspond to lines of slope one, leaving when $x' = 0$ and $t' = \tau$. They go until they intercept the ship's path.

When the (only) signal from the ship is emitted, it also travels along a null path. Since it is going in the opposite direction, this means it is a line of slope -1, passing through the emission point $x_c^{\mu\prime}$.

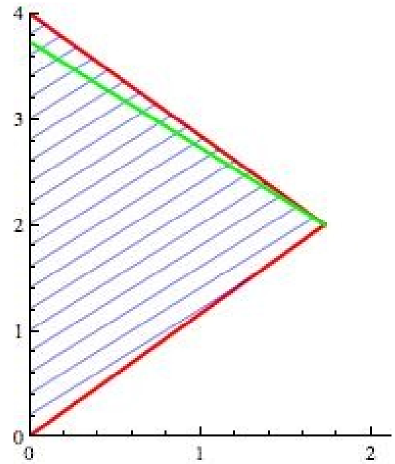


Figure 2: A spacetime diagram in the Earth's frame, using the modified inputs, where $\gamma = 2$ and signals are sent from the Earth every 0.2 years. The horizontal axis is $x'$, and the vertical is $t'$. The red lines correspond to the path the ship takes. The blue lines correspond to signals sent from the Earth, and the green line corresponds to the signal sent from the ship.

To find the times when the flashes from Earth get to the ship, we just find the $x'$ position where the line $t' = x' + \tau$ intercepts the ship's path. Transforming these to the ship's frame is a little more difficult, since there are actually *three* frames of reference: the "Earth" frame ($F_1$), the "ship going to the casino" frame ($F_2$), and the "ship returning from the casino" frame ($F_3$). For all of the events with $t' < \gamma$, we want to transform between frames $F_1$ and $F_2$. For all the rest, we need to transform between $F_1$ and $F_3$.

For the first transformation, we are going from a frame that is still to a frame moving at $\beta$, which is the opposite of what we usually do. This means we need to do the transformation using $-\beta$:

$$\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} t' \\ x' \end{pmatrix}$$

For the second transformation, the ship is traveling back toward the earth, so we use $\beta$. However, we also need to remember that the origins of $F_1$ and $F_3$ never coincided! We need to subtract off the origin of the $F_3$ frame from the $x'$ position used in the transformation. We can find this origin by finding where the ship would have been at $t' = 0$ in frame $F_1$ if it had started from the origin in $F_3$. Extending the line from the returning trip back to the $x'$ axis gives us $x' = 2\gamma\beta$. Thus, the transformation we need to use is

$$\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} t' \\ x' - 2\gamma\beta \end{pmatrix}$$

Now we want to find the time that the flash from the ship reaches the Earth's frame. This is easier: all we need to do is find, in the Earth's frame, where the line with slope -1 passing through the point $x_c^{\mu\prime}$ gets to $x' = 0$. Solving, this is $t' = \gamma(1 + \beta)$.

Using the modified parameters to the problem (the ones corresponding to the diagram), the numerical results are shown in Tables 1, 2, and 3. Using the parameters given in the problem ($\gamma = 100$, signals sent every year), we get the times shown in Tables 4, 5, 6.

| | | | |
|---|---|---|---|
| 1.49282 | 2.07077 | 2.17795 | 2.28513 |
| 2.39230 | 2.49948 | 2.60666 | 2.71384 |
| 2.82102 | 2.92820 | 3.03538 | 3.14256 |
| 3.24974 | 3.35692 | 3.46410 | 3.57128 |
| 3.67846 | 3.78564 | 3.89282 | |

Table 1: With modified inputs, the times when the ship receives the signals from Earth (Earth frame). All units are in years.

| | | | |
|---|---|---|---|
| 0.74641 | 1.03538 | 1.08897 | 1.14256 |
| 1.19615 | 1.24974 | 1.30333 | 1.35692 |
| 1.41051 | 1.46410 | 1.51769 | 1.57128 |
| 1.62487 | 1.67846 | 1.73205 | 1.78564 |
| 1.83923 | 1.89282 | 1.94641 | |

Table 2: With modified inputs, the times when the ship receives the signals from Earth (Ship frame). All units are in years.

| | |
|---|---|
| Earth frame: | 3.73205 |
| Ship frame: | 1.46410 |

Table 3: With modified inputs, the times when the Earth receives the signals from the ship. All units are in years.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 100.498 | 100.998 | 101.498 | 101.998 | 102.498 | 102.998 | 103.498 | 103.998 | 104.498 | 104.998 |
| 105.498 | 105.998 | 106.498 | 106.998 | 107.498 | 107.998 | 108.498 | 108.998 | 109.498 | 109.998 |
| 110.498 | 110.998 | 111.498 | 111.998 | 112.498 | 112.998 | 113.498 | 113.998 | 114.498 | 114.998 |
| 115.498 | 115.998 | 116.498 | 116.998 | 117.498 | 117.998 | 118.498 | 118.998 | 119.498 | 119.998 |
| 120.498 | 120.998 | 121.498 | 121.998 | 122.498 | 122.998 | 123.498 | 123.998 | 124.498 | 124.998 |
| 125.498 | 125.998 | 126.498 | 126.998 | 127.498 | 127.998 | 128.498 | 128.998 | 129.498 | 129.998 |
| 130.498 | 130.998 | 131.498 | 131.998 | 132.498 | 132.998 | 133.498 | 133.998 | 134.498 | 134.998 |
| 135.498 | 135.998 | 136.498 | 136.998 | 137.498 | 137.998 | 138.498 | 138.998 | 139.498 | 139.998 |
| 140.499 | 140.999 | 141.499 | 141.999 | 142.499 | 142.999 | 143.499 | 143.999 | 144.499 | 144.999 |
| 145.499 | 145.999 | 146.499 | 146.999 | 147.499 | 147.999 | 148.499 | 148.999 | 149.499 | 149.999 |
| 150.499 | 150.999 | 151.499 | 151.999 | 152.499 | 152.999 | 153.499 | 153.999 | 154.499 | 154.999 |
| 155.499 | 155.999 | 156.499 | 156.999 | 157.499 | 157.999 | 158.499 | 158.999 | 159.499 | 159.999 |
| 160.499 | 160.999 | 161.499 | 161.999 | 162.499 | 162.999 | 163.499 | 163.999 | 164.499 | 164.999 |
| 165.499 | 165.999 | 166.499 | 166.999 | 167.499 | 167.999 | 168.499 | 168.999 | 169.499 | 169.999 |
| 170.499 | 170.999 | 171.499 | 171.999 | 172.499 | 172.999 | 173.499 | 173.999 | 174.499 | 174.999 |
| 175.499 | 175.999 | 176.499 | 176.999 | 177.499 | 177.999 | 178.499 | 178.999 | 179.499 | 179.999 |
| 180.500 | 181.000 | 181.500 | 182.000 | 182.500 | 183.000 | 183.500 | 184.000 | 184.500 | 185.000 |
| 185.500 | 186.000 | 186.500 | 187.000 | 187.500 | 188.000 | 188.500 | 189.000 | 189.500 | 190.000 |
| 190.500 | 191.000 | 191.500 | 192.000 | 192.500 | 193.000 | 193.500 | 194.000 | 194.500 | 195.000 |

Table 4: With original inputs, times when the ship receives the signals from Earth (Earth frame). All units are in years.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1.00498 | 1.00998 | 1.01498 | 1.01998 | 1.02498 | 1.02998 | 1.03498 | 1.03998 | 1.04498 | 1.04998 |
| 1.05498 | 1.05998 | 1.06498 | 1.06998 | 1.07498 | 1.07998 | 1.08498 | 1.08998 | 1.09498 | 1.09998 |
| 1.10498 | 1.10998 | 1.11498 | 1.11998 | 1.12498 | 1.12998 | 1.13498 | 1.13998 | 1.14498 | 1.14998 |
| 1.15498 | 1.15998 | 1.16498 | 1.16998 | 1.17498 | 1.17998 | 1.18498 | 1.18998 | 1.19498 | 1.19998 |
| 1.20498 | 1.20998 | 1.21498 | 1.21998 | 1.22498 | 1.22998 | 1.23498 | 1.23998 | 1.24498 | 1.24998 |
| 1.25498 | 1.25998 | 1.26498 | 1.26998 | 1.27498 | 1.27998 | 1.28498 | 1.28998 | 1.29498 | 1.29998 |
| 1.30498 | 1.30998 | 1.31498 | 1.31998 | 1.32498 | 1.32998 | 1.33498 | 1.33998 | 1.34498 | 1.34998 |
| 1.35498 | 1.35998 | 1.36498 | 1.36998 | 1.37498 | 1.37998 | 1.38498 | 1.38998 | 1.39498 | 1.39998 |
| 1.40499 | 1.40999 | 1.41499 | 1.41999 | 1.42499 | 1.42999 | 1.43499 | 1.43999 | 1.44499 | 1.44999 |
| 1.45499 | 1.45999 | 1.46499 | 1.46999 | 1.47499 | 1.47999 | 1.48499 | 1.48999 | 1.49499 | 1.49999 |
| 1.50499 | 1.50999 | 1.51499 | 1.51999 | 1.52499 | 1.52999 | 1.53499 | 1.53999 | 1.54499 | 1.54999 |
| 1.55499 | 1.55999 | 1.56499 | 1.56999 | 1.57499 | 1.57999 | 1.58499 | 1.58999 | 1.59499 | 1.59999 |
| 1.60499 | 1.60999 | 1.61499 | 1.61999 | 1.62499 | 1.62999 | 1.63499 | 1.63999 | 1.64499 | 1.64999 |
| 1.65499 | 1.65999 | 1.66499 | 1.66999 | 1.67499 | 1.67999 | 1.68499 | 1.68999 | 1.69499 | 1.69999 |
| 1.70499 | 1.70999 | 1.71499 | 1.71999 | 1.72499 | 1.72999 | 1.73499 | 1.73999 | 1.74499 | 1.74999 |
| 1.75499 | 1.75999 | 1.76499 | 1.76999 | 1.77499 | 1.77999 | 1.78499 | 1.78999 | 1.79499 | 1.79999 |
| 1.80500 | 1.81000 | 1.81500 | 1.82000 | 1.82500 | 1.83000 | 1.83500 | 1.84000 | 1.84500 | 1.85000 |
| 1.85500 | 1.86000 | 1.86500 | 1.87000 | 1.87500 | 1.88000 | 1.88500 | 1.89000 | 1.89500 | 1.90000 |
| 1.90500 | 1.91000 | 1.91500 | 1.92000 | 1.92500 | 1.93000 | 1.93500 | 1.94000 | 1.94500 | 1.95000 |

Table 5: With original inputs, times when the ship receives the signals from Earth (Ship frame). All units are in years.

| | |
|---|---|
| Earth frame: | 199.995 |
| Ship frame: | 1.49999 |

Table 6: With original inputs, times when the Earth receives the signals from the ship. All units are in years.